M321 0S

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

Partial Differential Equations of Applied Mathematics Unit 0 (Solutions)

Some Useful Techniques

Solutions

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1.1

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0S.1 ORDINARY DERIVATIVES

0S.1.1 Differentiation

Solution to Exercise 1

(i) sin′ = cos (see *Unit M100 12*) and so

$$\frac{d}{dx}(\sin x) = \boxed{\cos x}.$$

(ii) The composite function rule is

$$(f \circ g)' = (f' \circ g) \times g'$$

(see *Unit M100 12*). Here $f = \exp$ and g is $x \longmapsto (x - a)$, or just $(x - a)$ for short. Omitting the small circles as well, for brevity, we get

$$(\exp(x - a))' = \exp'(x - a) \times (x - a)'$$

$$= \exp(x - a) \times 1,$$

since exp′ = exp (see *Unit M100 12*),

$$= \exp(x - a).$$

We could alternatively have written the composite function rule in Leibniz notation,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx},$$

(often called the *chain rule* for functions of one variable), where $z(y) = \exp y$ and $y(x) = x - a$. Then

$$\frac{dz}{dx} = \exp y \times 1$$

$$= \exp(x - a).$$

In either case we get

$$\frac{d}{dx}\Big(\exp(x-a)\Big) = \boxed{\exp(x - a)}.$$

(iii) We can use either of the methods in part (ii) above. Using the Leibniz chain rule with $z(y) = \sqrt{y}$ and $y(x) = \sin x$,

$$\frac{dz}{dx} = \tfrac{1}{2}y^{-\frac{1}{2}} \times \cos x,$$

so that

$$\frac{d}{dx}\Big(\sqrt{\sin x}\Big) = \boxed{\frac{\cos x}{2\sqrt{\sin x}}}.$$

(iv) The definitions we need are

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

from which we see that the derived function of cosh is sinh (*not* minus sinh; the analogy between trigonometric and hyperbolic functions is imperfect). Applying the chain rule as above with $z(y) = \cosh y$, $y(x) = x^2$,

$$\frac{dz}{dx} = \sinh y \times 2x,$$

so that

$$\frac{d}{dx}(\cosh x^2) = \boxed{2x \sinh x^2}.$$

Solution to Exercise 2

Using the chain rule with $z(y) = 1/y$, $y(x) = \cos x$,

$$\begin{aligned}\frac{dz}{dx} &= -\frac{1}{y^2} \times (-\sin x),\\ &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

so that

$$\frac{d}{dx}(\sec x) = \boxed{\tan x \sec x}.$$

If $h = f/g$, then

$$h' = \frac{f'g - fg'}{g^2}.$$

This is the form of the quotient rule stated in *Unit M100 12.* In Leibniz notation, if $u = u(x)$ and $v = v(x)$ then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Putting $u(x) = \sin x$, $v(x) = \cos x$,

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{1}{\cos^2 x},\end{aligned}$$

so that

$$\frac{d}{dx}(\tan x) = \boxed{\sec^2 x}.$$

This could also be written in other equivalent forms, such as $1 + \tan^2 x$.

Solution to Exercise 3

Since

$$x = \tfrac{1}{2}at^2,$$

we have

$$\begin{aligned}\frac{dx}{dt} &= at\\ &= \sqrt{2ax},\end{aligned}$$

so that

$$v(t) = \boxed{at},$$

$$v(x) = \boxed{\sqrt{2ax}}.$$

If $t = 4$, then $v = 4a$, so that

$$v(t = 4) = \boxed{4a}.$$

If $x = 4$, then $v = \sqrt{8a}$, so that

$$v(x = 4) = \boxed{\sqrt{8a}}.$$

$v(t = x)$ is the expression for $v(t)$ with x replacing t,

$$v(t = x) = \boxed{ax}.$$

$v(x = t)$ is the expression for $v(x)$ with t replacing x,

$$v(x = t) = \boxed{\sqrt{2at}}.$$

$v|_{x=3}$ is the same as $v(x = 3)$, that is

$$v|_{x=3} = \boxed{\sqrt{6a}}.$$

The derivative of v with respect to t is $d(at)/dt$, so that

$$\frac{dv}{dt} = \boxed{a}.$$

The derivative of v with respect to x is $d(\sqrt{2ax})/dx$, which is

$$\sqrt{2a}\frac{d}{dx}(x^{\frac{1}{2}}) = \tfrac{1}{2}\sqrt{2a}x^{-\frac{1}{2}},$$

or

$$\frac{dv}{dx} = \boxed{\sqrt{\frac{a}{2x}}}.$$

If $x = 1$, then $dv/dx = \sqrt{(a/2)}$, or

$$\left.\frac{dv}{dx}\right|_{x=1} = \boxed{\sqrt{\frac{a}{2}}}.$$

If $t = t_0$, then $x = \frac{1}{2}at_0^2$, so that

$$\left.\frac{dv}{dx}\right|_{t=t_0} = \left.\frac{dv}{dx}\right|_{x=\frac{1}{2}at_0^2}$$

$$= \sqrt{\frac{a}{2(\frac{1}{2}at_0^2)}},$$

giving

$$\left.\frac{dv}{dx}\right|_{t=t_0} = \boxed{\frac{1}{t_0}}.$$

Solution to Exercise 4

From above,

$$\frac{dx}{dt} = \boxed{3t^2}.$$

We have $v(t) = 3t^2$, $v(x) = 3x^{\frac{2}{3}}$, so that

$$\frac{dv}{dt} = \boxed{6t},$$

$$\frac{dv}{dx} = \boxed{2x^{-\frac{1}{3}}}.$$

Expressing dv/dx in terms of t,

$$\frac{dv}{dx} = \frac{2}{t},$$

giving

$$\frac{dv}{dx}\frac{dx}{dt} = \boxed{6t}.$$

$$\text{Is} \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}? \boxed{\text{Yes}}$$

Solution to Exercise 5

Since $x = \frac{1}{2}at^2$ in Exercise 3,

$$x(t) \text{ is } t \longmapsto \boxed{\tfrac{1}{2}at^2}.$$

Differentiating with respect to t,

$$\frac{dx}{dt} = \boxed{at}.$$

From Exercise 3,

$$\frac{dv}{dt} = \boxed{a},$$

and

$$\frac{dv}{dx} = \boxed{\sqrt{\frac{a}{2x}}}.$$

Expressing this in terms of t,

$$\frac{dv}{dx} = \frac{1}{t},$$

so that

$$\frac{dv}{dx}\frac{dx}{dt} = \boxed{a}.$$

$$\text{Is} \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}? \boxed{\text{Yes}}$$

Solution to Exercise 6

If $z = z(y)$, $y = y(x)$ and $x = x(t)$, we know that the chain rule gives both

$$\frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dt}$$

and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Combining these two,

$$\frac{dz}{dt} = \boxed{\frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dt}}.$$

Put $z = \ln y$, $y = \cos x$, $x = e^{2t}$. Then

$$\frac{dz}{dy} = \frac{1}{y} = \frac{1}{\cos x} = \frac{1}{\cos e^{2t}},$$

$$\frac{dy}{dx} = -\sin x = -\sin e^{2t},$$

$$\frac{dx}{dt} = 2e^{2t},$$

so that

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{\cos e^{2t}}(-\sin e^{2t})2e^{2t}\\ &= -2e^{2t}\tan e^{2t}.\end{aligned}$$

Thus,

$$\frac{d}{dt}(\ln \cos e^{2t}) = \boxed{-2e^{2t}\tan e^{2t}}.$$

Solution to Exercise 7

If $y(x) = \arctan x$, then

$$x(y) = \boxed{\tan y};$$

hence, by Exercise 2,

$$\frac{dx}{dy}(y) = \boxed{\sec^2 y};$$

hence, by the chain rule,

$$\frac{dy}{dx}(y) = \boxed{\cos^2 y}.$$

Since $\sec^2 y = 1 + \tan^2 y$,

$$\cos^2 y = \frac{1}{1 + x^2};$$

thus

$$\frac{dy}{dx}(x) = \boxed{\frac{1}{1 + x^2}},$$

so that

$$\frac{d}{dx}(\arctan x) = \boxed{\frac{1}{1+x^2}}.$$

0S.1.2 Integration

Solution to Exercise

The methods described below are all based very firmly on the M201 supplementary handbook, *Techniques of Integration* (***TI***). If you know a quicker method, so much the better. The constant of integration is omitted in each indefinite integral.

(i) $\int x\,dx$. The integrand x is a polynomial function of x, so we can use the formula in section III.1 on page 8 of ***TI***, with $a_1 = 1$, and $a_i = 0$ for all $i \neq 1$, to get

$$\int x\,dx = \boxed{\tfrac{1}{2}x^2}.$$

(ii) $\int t^2\,dt$. The integrand is a polynomial function of t. We use the same formula as in part (i), but on this occasion t replaces x and we have $a_2 = 1$ and $a_i = 0$ for all $i \neq 2$. Thus

$$\int t^2\,dt = \boxed{\tfrac{1}{3}t^3}.$$

(iii) $\int e^{-x}\,dx$. The integrand involves an exponential function, so we look at section III.5 on page 14 of ***TI***. Formula 5.2(i) with $a = -1$, $b = 0$ gives

$$\int e^{-x}\,dx = \boxed{-e^{-x}}.$$

Alternatively formula 5.2(iii) with $a = -1$, $n = 0$ (or even formula 5.2(vi) with $a = e^{-1}$) gives the same answer.

(iv) $\int \frac{1}{1+x^2}dx$. The integrand is a rational function of x, so we can use section III.2 on page 8 of ***TI***. The first relevant paragraph is 2.3, with $p = 0$, $q = 1$, $b = 0$, $c = 1$ (and note that $b^2 < 4c$; if this had not been so we would have had to go on to paragraph 2.4). The recommended substitution $t = x + \frac{1}{2}b$ is just $t = x$, i.e. we can proceed to the next step immediately. Using the second of the two given standard integrals, with $c = 1$ and x in place of t, we get

$$\int \frac{1}{1+x^2}dx = \boxed{\arctan x}.$$

The arctan function was mentioned in Exercise 7 of subsection 0.1.1, and you may have been able to answer this question by using the result of that exercise. A graph of the arctan function is shown on page 7 of ***TI***.

(v) $\int \frac{1}{4+x^2}dx$. As in part (iv), we are led to paragraph 2.3 in part III of ***TI***. This time we have $p = 0$, $q = 1$, $b = 0$, $c = 4$, and so the second standard integral at the top of page 9 gives

$$\int \frac{1}{4+x^2}dx = \boxed{\tfrac{1}{2}\arctan \tfrac{1}{2}x}.$$

(vi) $\int \sin \alpha x \sin \beta x \, dx$. The integrand involves trigonometric functions, so we turn to section III.5 on page 14 of ***TI***. Using the first formula in paragraph 5.1, with α, β replacing a, b respectively, brings the integral to the form

$$\tfrac{1}{2}\int \cos(\alpha - \beta)x \, dx - \tfrac{1}{2}\int \cos(\alpha + \beta)x \, dx.$$

The next relevant formula is 5.2(i) with $a = 0$ and b put equal to $\alpha - \beta$ and $\alpha + \beta$ in turn. This gives

$$\int \sin \alpha x \sin \beta x \, dx = \boxed{\frac{1}{2}\left[\frac{\sin(\alpha - \beta)x}{\alpha - \beta} - \frac{\sin(\alpha + \beta)x}{\alpha + \beta}\right]}.$$

(vii) $\int_0^\pi \sin mx \sin nx \, dx$. We have two cases to consider, depending upon whether or not m and n are equal. Suppose $m \neq n$. We then have from the answer to part (vi), with m, n replacing α, β repectively,

$$\begin{aligned}\int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2}\left[\frac{\sin(m - n)x}{m - n} - \frac{\sin(m + n)x}{m + n}\right]_0^\pi \\ &= 0,\end{aligned}$$

since the sine of any integer multiple of π is zero. Suppose next that $m = n$. Our integrand is now $\sin^2 nx$, and using the first formula in paragraph 5.1 (section III.5 on page 14 of ***TI***) with $a = b = n$,

$$\begin{aligned}\int_0^\pi \sin^2 nx \, dx &= \tfrac{1}{2}\int_0^\pi (1 - \cos 2nx) \, dx \\ &= \tfrac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_0^\pi,\end{aligned}$$

having used formula 5.2(i) with $a = 0$, $b = 2n$ to integrate the second term of the integrand. Thus

$$\int_0^\pi \sin^2 nx \, dx = \tfrac{1}{2}\pi,$$

so that

$$\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} \boxed{\tfrac{1}{2}\pi} & \text{if } m = n, \\ \boxed{0} & \text{if } m \neq n. \end{cases}$$

(viii) $\int_0^\pi \cos x \sin nx \, dx$. We again start with paragraph 5.1 of section III.5 on page 14 of ***TI***. Using the third formula in this paragraph, with $a = n, b = 1$, our integral becomes

$$\tfrac{1}{2}\int_0^\pi \sin (n - 1)x \, dx + \tfrac{1}{2}\int_0^\pi \sin (n + 1)x \, dx.$$

We then use formula 5.2(ii) with $a = 0$ and b equal to $n - 1, n + 1$ in turn to get

$$\int_0^\pi \cos x \sin nx \, dx = \frac{1}{2}\left[-\frac{\cos(n - 1)x}{n - 1} - \frac{\cos(n + 1)x}{n + 1}\right]_0^\pi$$

provided that $n \neq 1$ (if $n = 1$, we see that the first integrand above is zero, so that the first term in the square bracket should be replaced by zero in this case). Since $\cos m\pi = (-1)^m$ if m is an integer,

$$\int_0^\pi \cos x \sin nx \, dx = \begin{cases} \boxed{\dfrac{1}{n - 1} + \dfrac{1}{n + 1}} & \text{if } n \text{ is even}, \\ \boxed{0} & \text{if } n \text{ is odd}. \end{cases}$$

(ix) $\int_0^\pi x \sin nx\, dx$. We can perform the integration by using formula 5.2(v) of section III.5 in *TI* with n replaced by 1 and a replaced by n. This gives

$$\int_0^\pi x \sin nx\, dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^\pi$$
$$= -\frac{\pi}{n}\cos n\pi,$$

so that

$$\int_0^\pi x \sin nx\, dx = \boxed{(-1)^{n+1}\frac{\pi}{n}}.$$

(x) $\int_{-\pi}^\pi x^2 \cos nx\, dx$. Using formula 5.2(iv) of section III.5 with n replaced by 2 and a replaced by n,

$$\int_{-\pi}^\pi x^2 \cos nx\, dx = \left[\left(\frac{x^2}{n} - \frac{2}{n^3}\right)\sin nx + \frac{2x}{n^2}\cos nx\right]_{-\pi}^\pi$$
$$= \frac{2\pi}{n^2}\cos n\pi - \frac{2(-\pi)}{n^2}\cos(-n\pi)$$
$$= \frac{4\pi}{n^2}\cos n\pi,$$

so that

$$\int_{-\pi}^\pi x^2 \cos nx\, dx = \boxed{4(-1)^n\frac{\pi}{n^2}}.$$

(xi) $\int_{-\pi}^\pi f(x)\sin nx\, dx$. Since $f(x) = 0$ for $-\pi \leqslant x \leqslant 0$ and $f(x) = 1$ for $0 < x \leqslant \pi$, we break our integral up into two parts:

$$\int_{-\pi}^\pi f(x)\sin nx\, dx = \int_{-\pi}^0 0 \times \sin nx\, dx + \int_0^\pi 1 \times \sin nx\, dx$$
$$= \int_0^\pi \sin nx\, dx$$
$$= \left[-\frac{1}{n}\cos nx\right]_0^\pi,$$

having used formula 5.2(ii) with $a = 0$, $b = n$. Since

$$\left[-\frac{1}{n}\cos nx\right]_0^\pi = -\frac{1}{n}\cos n\pi + \frac{1}{n},$$

it follows that

$$\int_{-\pi}^\pi f(x)\sin nx\, dx = \boxed{\frac{1}{n}(1-(-1)^n)}.$$

(xii) $\int_0^1 f(x)\sin n\pi x\, dx$. Since $f(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$ and $f(x) = 1 - x$ for $\frac{1}{2} \leqslant x \leqslant 1$, we break our integral into two parts:

$$\int_0^1 f(x)\sin n\pi x\, dx = \int_0^{\frac{1}{2}} f(x)\sin n\pi x\, dx + \int_{\frac{1}{2}}^1 f(x)\sin n\pi x\, dx$$
$$= \int_0^{\frac{1}{2}} x \sin n\pi x\, dx + \int_{\frac{1}{2}}^1 (1 - x)\sin n\pi x\, dx.$$

We see by putting $n = 1$, $a = n\pi$ in formula 5.2(v) in section III.5 on page 14 of ***TI*** that

$$\int x \sin n\pi x \, dx = -\frac{x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x.$$

By putting $n = 0$, $a = n\pi$ in the same formula,

$$\int \sin n\pi x \, dx = -\frac{1}{n\pi}\cos n\pi x.$$

Thus,

$$\begin{aligned}\int_0^1 f(x)\sin n\pi x \, dx &= \left[-\frac{x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^{\frac{1}{2}} \\ &\quad + \left[-\frac{1}{n\pi}\cos n\pi x + \frac{x}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]_{\frac{1}{2}}^1 \\ &= -\frac{1}{2n\pi}\cos \tfrac{1}{2}n\pi + \frac{1}{n^2\pi^2}\sin \tfrac{1}{2}n\pi \\ &\quad + \frac{1}{2n\pi}\cos \tfrac{1}{2}n\pi + \frac{1}{n^2\pi^2}\sin \tfrac{1}{2}n\pi \\ &= \frac{2}{n^2\pi^2}\sin \tfrac{1}{2}n\pi.\end{aligned}$$

Now $\sin \frac{1}{2}n\pi$ is zero if n is even. If n is odd, $\sin \frac{1}{2}n\pi = 1$ if $\frac{1}{2}(n-1)$ is even and $\sin \frac{1}{2}n\pi = -1$ if $\frac{1}{2}(n-1)$ is odd.
Therefore,

$$\int_0^1 f(x)\sin n\pi x \, dx = \begin{cases} \boxed{0} & \text{if } n \text{ is even,} \\ \boxed{(-1)^{\frac{1}{2}(n-1)}\dfrac{2}{n^2\pi^2}} & \text{if } n \text{ is odd.}\end{cases}$$

0S.1.3 Differential Equations of First Order and Degree

Solution to Exercise

(i) Integrating both sides we obtain

$$x(t) = \boxed{\tfrac{1}{3}t^3 + A}.$$

(ii) Separating variables (see page 20 of *Unit M100 24* or section 4.2 of ***SS*** if necessary),

$$\int e^{-x}\,dx = \int dt + A,$$

or

$$-e^{-x} = t + A.$$

Therefore

$$x(t) = \boxed{-\ln(-t - A)} \qquad t \in (-\infty, -A).$$

Notice that in this case no solution has as domain the whole of R. This commonly happens with nonlinear differential equations, for which the simple uniqueness theorem studied in *Units M201 4* and *M201 33* may not apply.

(iii) Separating variables,

$$\int x\,dx = \int t\,dt + C,$$

giving

$$\tfrac{1}{2}x^2 = \tfrac{1}{2}t^2 + C,$$

so that

$$x(t) = \boxed{\pm\sqrt{t^2 + A}},$$

where $A = 2C$, and $t \in R$ if $A \geqslant 0$, $t \in (\sqrt{-A}, \infty) \cup (-\infty, -\sqrt{-A})$ if $A < 0$.

(iv) Separating variables,

$$\int \frac{1}{1 + x^2}\,dx = \int dt + A,$$

and using part (iv) of the Exercise in subsection 0.1.2 this gives

$$\arctan x = t + A,$$

or

$$x(t) = \boxed{\tan(t + A)} \quad t \in (-A - \tfrac{1}{2}\pi, -A + \tfrac{1}{2}\pi).$$

(v) Separating variables,

$$\int \frac{1}{4 + x^2}\,dx = \int dt + C,$$

so that using part (v) of the Exercise in subsection 0.1.2 gives

$$\tfrac{1}{2}\arctan\tfrac{1}{2}x = t + C,$$

or

$$x(t) = \boxed{2\tan(2t + A)} \quad t \in (-\tfrac{1}{2}A - \tfrac{1}{4}\pi, -\tfrac{1}{2}A + \tfrac{1}{4}\pi),$$

where $A = 2C$.

(vi) The equation can be written as

$$\frac{dx}{dt} - \frac{1}{t}x = 0,$$

so that an integrating factor is

$$\exp\int\left(-\frac{1}{t}\right)dt = \exp(-\ln t) = \frac{1}{t}$$

(see lines 6 to 10 on page 97 of K if necessary). Multiplying through by this factor, we see that the equation can be put in the form

$$\frac{d}{dt}\left(\frac{x}{t}\right) = 0,$$

which has solution

$$x(t) = \boxed{At}.$$

0S.1.4 Differential Equations of First Order and Second Degree

Solution to Exercise

(i) Factorizing the equation gives

$$\left(2\frac{dx}{dt} - 1\right)\left(\frac{dx}{dt} + 1\right) = 0,$$

so that

$$\frac{dx}{dt} = \tfrac{1}{2} \text{ or } -1.$$

(Alternatively you can use the general quadratic equation formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to achieve this result.)
Integrating,

$$x(t) = \boxed{-t + A} \quad \text{or} \quad \boxed{\tfrac{1}{2}t + A}.$$

(ii) Factorizing gives

$$\left(\frac{dx}{dt} - e^x\right)\left(\frac{dx}{dt} - t^2\right) = 0,$$

so that

$$\frac{dx}{dt} = e^x \quad \text{or} \quad t^2.$$

Hence from parts (i) and (ii) of the Exercise in subsection 0.1.3,

$$x(t) = \boxed{-\ln(-t - A)} \quad \text{or} \quad \boxed{\tfrac{1}{3}t^3 + A},$$

with the restriction that $t \in (-\infty, -A)$ in the first case.

0S.2 FIRST-ORDER PARTIAL DIFFERENTIATION

0S.2.1 First-Order Partial Derivatives

Solution to Exercise 1

(i) $$\frac{\partial u}{\partial x}(x, y) = \boxed{2x}.$$

$$\frac{\partial u}{\partial y}(x, y) = \boxed{2y}.$$

(ii) $$\frac{\partial u}{\partial x}(x, y) = \boxed{y^2 e^{xy}}.$$

$$\frac{\partial u}{\partial y}(x, y) = \boxed{(1 + xy)e^{xy}}.$$

(iii) $$\frac{\partial u}{\partial x}(x, y) = \boxed{\cos(x - y)}.$$

$$\frac{\partial u}{\partial y}(x, y) = \boxed{-\cos(x - y)}.$$

(iv) $$\frac{\partial u}{\partial x}(x, y) = \boxed{f'(x - y)}.$$

$$\frac{\partial u}{\partial y}(x, y) = \boxed{-f'(x - y)}.$$

Solution to Exercise 2

$$\frac{\partial}{\partial x}[p(x + ct) + q(x - ct)] = \boxed{p'(x + ct) + q'(x - ct)},$$

$$\frac{\partial}{\partial t}[p(x + ct) + q(x - ct)] = \boxed{cp'(x + ct) - cq'(x - ct)}.$$

Solution to Exercise 3

(i) Suppose that G is a primitive of g, so that by the fundamental theorem of calculus, $G' = g$. Then

$$\int_0^{x+ct} g(\bar{x})d\bar{x} = \Big[\, G(\bar{x}) \,\Big]_0^{x+ct}$$
$$= G(x + ct) - G(0),$$

so that

$$\frac{\partial}{\partial x}\int_0^{x+ct} g(\bar{x})d\bar{x} = \frac{\partial}{\partial x}\Big[G(x + ct) - G(0)\Big]$$
$$= G'(x + ct),$$

or

$$\frac{\partial}{\partial x}\int_0^{x+ct} g(\bar{x})d\bar{x} = \boxed{g(x + ct)}.$$

(ii) This is the same as for part (i), with $x - ct$ replacing $x + ct$ throughout. Thus

$$\frac{\partial}{\partial x}\int_0^{x-ct} g(\bar{x})d\bar{x} = \boxed{g(x - ct)}.$$

(iii) Since

$$\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x} = \int_{0}^{x+ct} g(\bar{x})d\bar{x} - \int_{0}^{x-ct} g(\bar{x})d\bar{x},$$

we have from parts (i) and (ii)

$$\frac{\partial}{\partial x}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x} = \boxed{g(x+ct) - g(x-ct)}.$$

(iv) Defining G as before,

$$\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{\partial}{\partial t}\Big[G(x+ct) - G(x-ct)\Big].$$

Using the second result of Exercise 2 with $p = G$, $q = -G$,

$$\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \boxed{cg(x+ct) + cg(x-ct)}.$$

Solution to Exercise 4

If $u(x, t) = \sin(x^2 - t)$, put $t = 0$ to get $u(x, 0) = u|_{t=0}$:

$$u|_{t=0} = \boxed{\sin x^2}.$$

The partial derivative of u with respect to t is

$$\frac{\partial u}{\partial t}(x, t) = -\cos(x^2 - t),$$

so that

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \boxed{-\cos x^2}.$$

The partial derivative of u with respect to x is

$$\frac{\partial u}{\partial x}(x, t) = \boxed{2x\cos(x^2 - t)}.$$

Also,

$$u|_{x=0} = \boxed{-\sin t},$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = \boxed{2\cos(1 - t)}.$$

0S.2.2 Integration of Partial Derivatives

Solution to Exercise 1

(i) Since partial differentiation of a function of x and y with respect to x is performed by treating y as a constant,

$$\frac{\partial u}{\partial x} = \boxed{0}.$$

(ii) As in part (i),

$$\frac{\partial u}{\partial x} = \boxed{0}.$$

(iii) As in parts (i) and (ii),

$$\frac{\partial u}{\partial x} = \boxed{0}.$$

Solution to Exercise 2

The general solution is the function we differentiated in Exercise 1(iii):

$u(x, y) =$ $f(y)$, where f is an arbitrary function of one variable.

Solution to Exercise 3

The general solution is

$u(x, y) =$ $f(x)$, where f is an arbitrary function of one variable.

A brief justification:

For fixed x, consider

$$y \longmapsto u(x, y).$$

This has derived function

$$y \longmapsto \frac{\partial u}{\partial y}(x, y),$$

which is

$$y \longmapsto 0$$

by the differential equation. Hence $y \longmapsto u(x, y)$ is a constant function (regarding x as a fixed number) so that

$$\begin{aligned} u(x, y) &= u(x, 0) \\ &= f(x) \qquad \text{(say).} \end{aligned}$$

Solution to Exercise 4

(i) The general solution of the associated homogeneous equation is

$$u(x, y) = f(y)$$

for arbitrary f. If we regard y as fixed, a particular primitive of the function

$$x \longmapsto y$$

is

$$x \longmapsto xy,$$

so that the general solution we require is

$$u(x, y) = \boxed{xy + f(y)}.$$

(ii) Since for fixed x the function

$$y \longmapsto xy$$

has a primitive

$$y \longmapsto \tfrac{1}{2}xy^2,$$

the general solution is

$$u(x, y) = \boxed{\tfrac{1}{2}xy^2 + f(x)}.$$

(iii) Since for fixed t the function

$$x \longmapsto x^2 + t^2$$

has a primitive

$$x \longmapsto \tfrac{1}{3}x^3 + xt^2,$$

the general solution is

$$u(x, t) = \boxed{\tfrac{1}{3}x^3 + xt^2 + f(t)}.$$

(iv) Since for fixed ξ the function

$$\eta \longmapsto r(\eta)$$

has a primitive

$$\eta \longmapsto \int r(\eta)\, d\eta,$$

the general solution is

$$w(\xi, \eta) = \boxed{\int r(\eta)\, d\eta + f(\xi)}.$$

Solution to Exercise 5

(i) By the standard formula (see lines 6 to 10 on page 97 of *K*), an integrating factor for the ordinary differential equation is

$$\exp \int \left(-\frac{1}{x}\right) dx = \exp(-\ln x),$$

so that the integrating factor is

$$\boxed{\frac{1}{x}}.$$

(ii) Using this factor for the partial differential equation we obtain

$$\frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2} = 0,$$

or

$$\frac{\partial}{\partial x}\left(\frac{u(x, y)}{x}\right) = 0.$$

Thus

$$\frac{u(x, y)}{} = f(y),$$

and the general solution is

$$u(x, y) = \boxed{xf(y)}.$$

Solution to Exercise 6

(i) The integrating factor is

$$\exp\int\left(-\frac{1}{y}\right)dx = \boxed{\exp\left(-\frac{x}{y}\right)}.$$

(ii) The partial differential equation becomes

$$\frac{\partial}{\partial x}\left(u(x, y)\exp\left(-\frac{x}{y}\right)\right) = 0,$$

so that the general solution is

$$u(x, y) = \boxed{e^{x/y}f(y)}.$$

0S.2.3 Partial Differential Operators

Solution to Exercise 1

Since the operator is to be first-order, we are limited to first- and zeroth-order derivatives, and there can be no products of the first-order derivatives since this would contravene the linearity of L. Thus the most general form possible for L is

$$L = \boxed{a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} + c(x, y)},$$

for arbitrary functions of two variables a, b and c.

Solution to Exercise 2

(i) Since

$$\frac{\partial}{\partial x}(x + y)^2 = \frac{\partial}{\partial y}(x + y)^2 = 2(x + y),$$

it follows that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(x + y)^2 = \boxed{4(x + y)}.$$

(ii) Since

$$\frac{\partial}{\partial x}(x-y)^2 = -\frac{\partial}{\partial y}(x-y)^2 = 2(x-y),$$

we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(x-y)^2 = \boxed{\quad 0 \quad}.$$

(iii) Since

$$\frac{\partial}{\partial x}\sin(x-y) = -\frac{\partial}{\partial y}\sin(x-y) = \cos(x-y),$$

we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\sin(x-y) = \boxed{\quad 0 \quad}.$$

(iv) Since

$$\frac{\partial}{\partial x}F(x,y) = -\frac{\partial}{\partial y}F(x-y) \quad = F'(x-y),$$

we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)F(x-y) = \boxed{\quad 0 \quad}.$$

0S.2.4 Differentiation Along a Curve: the Chain Rule

Solution to Exercise 1

(i) Put

$$x = 2t,\ y = \sin t$$

into the potential energy $\phi = x^2 + y^3$ to get

$$\phi(t) = \boxed{\quad 4t^2 + \sin^3 t \quad}.$$

(ii) Differentiating this expression for $\phi(t)$ gives

$$\phi'(t) = \frac{d\phi}{dt} = \boxed{\quad 8t + 3\sin^2 t\cos t \quad}.$$

Solution to Exercise 2

(i) We have

$$z = x^2 + y^3,$$

and

$$x = \xi(t),\ y = \eta(t),$$

so that $z(t) = \boxed{\quad [\xi(t)]^2 + [\eta(t)]^3 \quad}.$

(ii) Differentiating with respect to t gives

$$z'(t) = \frac{dz}{dt} = \boxed{\quad 2\xi(t)\xi'(t) + 3[\eta(t)]^2\eta'(t) \quad}$$

Solution to Exercise 3

(i) Since $x = \xi(t)$, we have

$$\frac{dx}{dt} = \boxed{\xi'(t)}.$$

(ii) If $z = x^2 + y^3$, we have

$$\frac{\partial z}{\partial x} = \boxed{2x}.$$

(iii) We want $\partial z/\partial x$ evaluated at $x = \xi(t)$, $y = \eta(t)$. From part (ii) we have

$$\frac{\partial z}{\partial x}(\xi(t), \eta(t)) = \boxed{2\xi(t)}.$$

(iv) By proceeding as in parts (i), (ii) and (iii) we find that

$$\frac{dy}{dt} = \eta'(t),$$

$$\frac{\partial z}{\partial y}(\xi(t), \eta(t)) = 3[\eta(t)]^2,$$

so that at $(\xi(t), \eta(t))$ on the surface,

$$\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \boxed{2\xi(t)\xi'(t) + 3[\eta(t)]^2\eta'(t)}.$$

This is the same as the formula for dz/dt in Exercise 2(ii).

Solution to Exercise 4

(i) This is the case in Exercises 2 and 3 (if z is replaced by u). Since the solutions to Exercises 2(ii) and 3(iv) are identical, we see that the boxed formula holds in this case.

$\boxed{\text{Yes}}$

(ii) The result is also true in general; a demonstration is given in Appendix I of *Unit MST 282 7* on pages 34, 35.

$\boxed{\text{Yes}}$

Solution to Exercise 5

The analogue of the formula in Exercise 4 for the case where u is a function of three variables x, y, z is

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}}$$

Solution to Exercise 6

(i) We have

$$\frac{\partial u}{\partial x} = 2x = 2t^2, \qquad \frac{dx}{dt} = 2t,$$

$$\frac{\partial u}{\partial y} = 2y = 2(t-1) \qquad \frac{dy}{dt} = 1,$$

$$\frac{\partial u}{\partial z} = 2z = 2t^3, \qquad \frac{dz}{dt} = 3t^2,$$

so that

$$\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = (2t^2)(2t) + 2(t-1) + (2t^3)(3t^2),$$

giving

$$\frac{du}{dt} = \boxed{6t^5 + 4t^3 + 2t - 2}.$$

(ii) Writing

$$u = (t^2)^2 + (t-1)^2 + (t^3)^2$$

or

$$u(t) = \boxed{t^6 + t^4 + t^2 - 2t + 1},$$

we have

$$\frac{du}{dt} = \boxed{6t^5 + 4t^3 + 2t - 2}.$$

The two methods agree.

Solution to Exercise 7

(i) By definition we have

$$\textbf{grad } u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k},$$

so that by reference to the solution of Exercise 6,

$$\textbf{grad } u = \boxed{2(t^2\mathbf{i} + (t-1)\mathbf{j} + t^3\mathbf{k})}.$$

(ii) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

or

$$\frac{d\mathbf{r}}{dt} = \boxed{2t\mathbf{i} + \mathbf{j} + 3t^2\mathbf{k}}.$$

(iii)
$$\begin{aligned}\textbf{grad } u \cdot \frac{d\mathbf{r}}{dt} &= 2(t^2\mathbf{i} + (t-1)\mathbf{j} + t^3\mathbf{k}) \cdot (2t\mathbf{i} + \mathbf{j} + 3t^2\mathbf{k}) \\ &= 2(2t^3 + (t-1) + 3t^5),\end{aligned}$$

or

$$\textbf{grad } u \cdot \frac{d\mathbf{r}}{dt} = \boxed{6t^5 + 4t^3 + 2t - 2},$$

which is the same as the expression for du/dt derived in Exercise 6.

0S.2.5 Derivatives "Following the Motion"

Solution to Exercise 1

The velocity at any point x in the centre of the stream at time t is $v(x, t)$. If the motion of the leaf (which is the same as that of the water immediately beneath it) is given by $x = x(t)$, its velocity at time t will be $v(x(t), t)$. But we can also find this velocity by differentiating the function $x(t)$. Thus

$$\boxed{\frac{dx}{dt} = v(x(t), t)}\,.$$

Solution to Exercise 2

The temperature $T(t)$ at the leaf at time t will be equal to the temperature $T(x, t)$ at position x and time t, where $x = x(t)$. That is,

$$T(t) = \boxed{T(x(t), t)}\,.$$

Solution to Exercise 3

(i) Since

$$T(x, t) = b \ln x + c \sin t,$$

and

$$x(t) = at + k,$$

we have

$$T(x(t), t) = b \ln(at + k) + c \sin t.$$

Thus, from the solution to Exercise 2,

$$T(t) = \boxed{b \ln(at + k) + c \sin t}\,.$$

(ii) Differentiating this expression,

$$\frac{dT}{dt} = \boxed{\frac{ab}{at + k} + c \cos t}\,.$$

(The first term of $T(t)$ can be differentiated using the chain rule for functions of one variable,

$$\frac{d}{dt}(b \ln x) = \frac{d}{dx}(b \ln x)\frac{dx}{dt},$$

where $x = at + k$.)

Solution to Exercise 4

Consider the data of Exercise 3. Computing the quantities to be included in the general expression for dT/dt we see that in Exercise 3 these were

$$\frac{\partial T}{\partial x} = \frac{b}{x} = \frac{b}{at + k},$$

$$\frac{\partial T}{\partial t} = c \cos t,$$

and

$$\frac{dx}{dt} = a.$$

Comparing with the solution to Exercise 3, we see that we can write

$$\frac{dT}{dt} = \boxed{\frac{dx}{dt}\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t}}.$$

We could alternatively derive this result from the (boxed) chain rule in Exercise 4 of subsection 0.2.4 upon replacing u by T and y by t.

Solution to Exercise 5

The text of the exercise after the first sentence should read as follows:

"The symbol T on the left refers to the function $\boxed{T(t)}$ with domain $\boxed{R}$, describing the temperature at the leaf at time t. The symbol T on the right refers to the function $\boxed{T(x, t)}$, with domain $\boxed{R \times R}$, describing the temperature at position $\boxed{x}$ at time t."

Solution to Exercise 6

Replacing T by v in the solution to Exercise 4, we find

$$\frac{dv}{dt} = \boxed{\frac{dx}{dt}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}}.$$

Solution to Exercise 7

The operator we seek is d/dt (as it appears on the left-hand sides of the solutions to Exercises 4 and 6). If we remove the particular dependent variable (v or T) involved in either of those expressions, we are left with

$$\frac{d}{dt} = \boxed{\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{\partial}{\partial t}}.$$

It was shown in Exercise 1 that $dx/dt = v(x(t), t)$. Thus for the particular leaf, the operator can be written as

$$v(x(t), t)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

We abandon the particular spot occupied by the leaf in favour of an arbitrary point of the stream by replacing the specific function $x(t)$ by the coordinate x. Writing v for $v(x, t)$, we are left with the operator equation

$$\frac{d}{dt} = \boxed{v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}}.$$

0S.2.6 Change of Coordinates

Solution to Exercise 1

The expressions for x and y are

$$x = \boxed{r\cos\theta},$$

$$y = \boxed{r\sin\theta}.$$

Solution to Exercise 2

Since $T = f(x, y)$, we can use the formulae from Exercise 1 to get

$$T = \boxed{f(r\cos\theta, r\sin\theta)}.$$

Solution to Exercise 3

Along any curve for which θ = constant we have $T = f(x, y)$, where x and y are functions of r alone. Using the boxed formula

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

(from Exercise 4 of subsection 0.2.4) with T for u and r for t, we have

$$\frac{dT}{dr} = \frac{\partial T}{\partial x}\frac{dx}{dr} + \frac{\partial T}{\partial y}\frac{dy}{dr} \text{ along } \theta = \text{constant}.$$

For functions of two variables r and θ however, the operator "d/dr along θ = constant" is, by definition, the same as $\partial/\partial r$. We can therefore write the above as

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r}.$$

Since in this case

$$\frac{\partial x}{\partial r} = \cos\theta,$$

$$\frac{\partial y}{\partial r} = \sin\theta,$$

we have

$$\frac{\partial T}{\partial r} = \boxed{\cos\theta\frac{\partial T}{\partial x} + \sin\theta\frac{\partial T}{\partial y}}.$$

Similarly we can write

$$\frac{dT}{d\theta} = \frac{\partial T}{\partial x}\frac{dx}{d\theta} + \frac{\partial T}{\partial y}\frac{dy}{d\theta} \text{ along } r = \text{constant},$$

or

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta}.$$

Since

$$\frac{\partial x}{\partial \theta} = -r\sin\theta,$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta,$$

we arrive at

$$\frac{\partial T}{\partial \theta} = \boxed{-r\sin\theta\frac{\partial T}{\partial x} + r\cos\theta\frac{\partial T}{\partial y}}.$$

Solution to Exercise 4

Since $T = f(x, y)$, and $x = \alpha p + \beta q$, $y = \gamma p + \delta q$,

$$T = \boxed{f(\alpha p + \beta q, \gamma p + \delta q)}.$$

We obtain the expressions for $\partial T/\partial p$ and $\partial T/\partial q$ as before by using the boxed formula

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

(from Exercise 4 of subsection 0.2.4) with the total derivatives replaced by partial derivatives, T for u, and first p and then q in place of t:

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial p},$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial q}.$$

Since

$$\frac{\partial x}{\partial p} = \alpha, \frac{\partial y}{\partial p} = \gamma, \frac{\partial x}{\partial q} = \beta, \frac{\partial y}{\partial q} = \delta,$$

we have

$$\frac{\partial T}{\partial p} = \boxed{\alpha\frac{\partial T}{\partial x} + \gamma\frac{\partial T}{\partial y}};$$

$$\frac{\partial T}{\partial q} = \boxed{\beta\frac{\partial T}{\partial x} + \delta\frac{\partial T}{\partial y}}.$$

Solution to Exercise 5

We proceed as before. Since $T = f(x, y)$ and $x = x(\xi, \eta)$, $y = y(\xi, \eta)$,

$$T = \boxed{f(x(\xi, \eta), y(\xi, \eta))}.$$

The chain rule is used as in Exercises 3 and 4 to give

$$\frac{\partial T}{\partial \xi} = \boxed{\frac{\partial T}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \xi}},$$

$$\frac{\partial T}{\partial \eta} = \boxed{\frac{\partial T}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \eta}}.$$

Solution to Exercise 6

The symbols used here are differently arranged from those in the general case of Exercise 5. We are given $\xi = \xi(x, t)$, $\eta = \eta(x, t)$ (for particular functions ξ and η), and the corresponding chain rule is

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}.$$

Since

$$\xi(x, t) = x + ct,$$

$$\eta(x, t) = x - ct,$$

we have

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial x} = 1,$$

$$\frac{\partial \xi}{\partial t} = -\frac{\partial \eta}{\partial t} = c,$$

so that

$$\frac{\partial u}{\partial x} = \boxed{\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}},$$

$$\frac{\partial u}{\partial t} = \boxed{c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)}.$$

Solution to Exercise 7

We have $T(x, z) = T(x, y) = f(x, y)$ where $y = x + z$. Using the general form of the chain rule appearing in the last exercise, with u, t, ξ, η being replaced by $T(x, z) = f(x, y)$, z, x and y respectively,

$$\frac{\partial T}{\partial x}(x, z) = \frac{\partial f}{\partial x} + \frac{\partial y}{\partial x}\frac{\partial f}{\partial y},$$

$$\frac{\partial T}{\partial z}(x, z) = \frac{\partial y}{\partial z}\frac{\partial f}{\partial y}$$

(since $\partial x/\partial x = 1$ and $\partial x/\partial z = 0$). Thus

$$\frac{\partial T}{\partial x}(x, z) = \boxed{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}},$$

$$\frac{\partial T}{\partial z}(x, z) = \boxed{\frac{\partial f}{\partial y}},$$

or

$$\frac{\partial T}{\partial x}(x, z) = \boxed{\frac{\partial T}{\partial x}(x, y) + \frac{\partial T}{\partial y}(x, y)},$$

$$\frac{\partial T}{\partial z}(x, z) = \boxed{\frac{\partial T}{\partial y}(x, y)}.$$

Solution to Exercise 8

By putting u for T in the solution to Exercise 7 we see that in terms of the variables x, z the equation takes the form

$$\boxed{\frac{\partial u}{\partial x}(x, z)} = 0.$$

Solution to Exercise 9

We see from Exercise 2 of subsection 0.2.2 that the general solution of

$$\frac{\partial u}{\partial x}(x, z) = 0$$

is

$$u(x, z) = \boxed{f(z)},$$

where f is an arbitrary function of one variable. Since $y = x + z$, we have $z = y - x$, so that the general solution of

$$\frac{\partial u}{\partial x}(x, y) + \frac{\partial u}{\partial y}(x, y) = 0$$

is

$$u(x, y) = \boxed{f(y - x)},$$

which agrees with the result suggested by Exercise 2 of subsection 0.2.3.

Solution to Exercise 10

We are asked to restate the general form of the chain rule given in the solution to Exercise 6 (with t being replaced by y) in two different ways.

(i) As operator equations these are

$$\frac{\partial}{\partial x} = \boxed{\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}},$$

$$\frac{\partial}{\partial y} = \boxed{\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}}.$$

(ii) As a matrix equation, they can be expressed by

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} \end{bmatrix} = \left[\boxed{\begin{matrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{matrix}}\right] \begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\[2ex] \dfrac{\partial u}{\partial \eta} \end{bmatrix}.$$

Solution to Exercise 11

There are two methods of solution. The first involves performing the matrix multiplication explicitly and using the chain rule:

$$AB = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial x}{\partial \eta}\right) & \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial y}{\partial \eta}\right) \\[2ex] \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial x}{\partial \eta}\right) & \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial y}{\partial \eta}\right) \end{bmatrix},$$

Using the solutions to Exercise 10(i) we see that this is equal to

$$\begin{bmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial y}{\partial x} \\[2ex] \dfrac{\partial x}{\partial y} & \dfrac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that

$$AB = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

The second method is to note that corresponding to the matrix equation of Exercise 10(ii) giving $\partial u/\partial x$, $\partial u/\partial y$ in terms of $\partial u/\partial \xi$ and $\partial u/\partial \eta$, there will be a similar equation giving $\partial u/\partial \xi$, $\partial u/\partial \eta$ in terms of $\partial u/\partial x$ and $\partial u/\partial y$. The chain rule says that this equation is

$$\begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\[2ex] \dfrac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} \end{bmatrix}$$

and the 2×2 matrix here is just B. By combining this equation with that of Exercise 10(ii) we see immediately that A and B must be inverse to each other, so that AB is the identity matrix. This result is the analogue in two dimensions of the one-dimensional formula

$$\frac{dy}{dx}\frac{dx}{dy} = 1;$$

but note that, for example,

$$\frac{\partial x}{\partial \xi}\frac{\partial \xi}{\partial x} \neq 1$$

in general, since the $(1, 1)$ elements of two matrices inverse to each other are not normally reciprocals.

0S.3 SECOND-ORDER PARTIAL DIFFERENTIATION

0S.3.1 Second and Higher Order Partial Derivatives

Solution to Exercise 1

If $u(x, t) = \sin xt + \sin x \sin t + t$ then

$$\frac{\partial u}{\partial x} = \boxed{t \cos xt + \cos x \sin t}.$$

$$\frac{\partial u}{\partial t} = \boxed{x \cos xt + \sin x \cos t + 1}.$$

$$\frac{\partial^2 u}{\partial x \partial t} = \boxed{\cos xt - xt \sin xt + \cos x \cos t}.$$

$$\frac{\partial^2 u}{\partial t \partial x} = \boxed{\cos xt - xt \sin xt + \cos x \cos t}.$$

$$\frac{\partial^2 u}{\partial t^2} = \boxed{-x^2 \sin xt - \sin x \sin t}.$$

Solution to Exercise 2

$\boxed{\text{Yes}}$,

provided that the two mixed derivatives exist and are continuous. All the functions to be encountered in the course will satisfy these conditions.

Solution to Exercise 3

Since $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ commute, we have

$$\begin{aligned}
\frac{\partial^{2n} f}{\partial x^{2n}} &= \left(\frac{\partial^2}{\partial x^2}\right)^n f \\
&= \left(\frac{\partial^2}{\partial x^2}\right)^{n-1} \left(\frac{\partial^2}{\partial x^2}\right) f \\
&= \left(\frac{\partial^2}{\partial x^2}\right)^{n-1} \left(\frac{\partial^2}{\partial y^2}\right) f \\
&= \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2}{\partial x^2}\right)^{n-1} f \\
&= \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2}{\partial x^2}\right)^{n-2} \left(\frac{\partial^2}{\partial y^2}\right) f \\
&= \left(\frac{\partial^2}{\partial y^2}\right)^2 \left(\frac{\partial^2}{\partial x^2}\right)^{n-2} f,
\end{aligned}$$

and so on until all the $\partial^2/\partial x^2$ have been changed into $\partial^2/\partial y^2$. Hence the result is

$$\left(\frac{\partial^2}{\partial x^2}\right)^n f = \left(\frac{\partial^2}{\partial y^2}\right)^n f,$$

or

$$\frac{\partial^{2n} f}{\partial x^{2n}} = \boxed{\frac{\partial^{2n} f}{\partial y^{2n}}}.$$

Solution to Exercise 4

This time we have

$$\left(\frac{\partial^2}{\partial x^2}\right)f = \left(-\frac{\partial^2}{\partial y^2}\right)f,$$

and so, the same procedure as we used in the preceding exercise gives

$$\left(\frac{\partial^2}{\partial x^2}\right)^n f = \left(-\frac{\partial^2}{\partial y^2}\right)^n f,$$

which can be written as

$$\frac{\partial^{2n} f}{\partial x^{2n}} = \boxed{(-1)^n \frac{\partial^{2n} f}{\partial y^{2n}}}.$$

Solution to Exercise 5

Proceeding as before we find that

$$\frac{\partial^n f}{\partial y^n} = \boxed{\frac{\partial^{2n} f}{\partial x^{2n}}}.$$

Hence

$$\frac{\partial^2 f}{\partial y^2} - 3\frac{\partial^3 f}{\partial y \partial x^2} + 2\frac{\partial^4 f}{\partial x^4} = \frac{\partial^2 f}{\partial y^2} - 3\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + 2\frac{\partial^2 f}{\partial y^2},$$

giving

$$\frac{\partial^2 f}{\partial y^2} - 3\frac{\partial^3 f}{\partial y \partial x^2} + 2\frac{\partial^4 f}{\partial x^4} = \boxed{0}.$$

0S.3.2 Solving Simple Second-Order Partial Differential Equations

Solution to Exercise 1

Putting $\partial u/\partial y = p$, the equation becomes

$$\boxed{\frac{\partial p}{\partial x} = 0},$$

with solution

$$p(x, y) = \boxed{f(y)}$$

for an arbitrary function f. Since

$$\frac{\partial u}{\partial y}(x, y) = \boxed{f(y)}$$

has solution

$$u = \int f(y) + g(x),$$

the general solution of the original equation is

$$u(x, y) = \boxed{\int f(y) + g(x)}$$

where f and g are arbitrary functions of one variable. Since the equation we started from is symmetric in x and y we could equally well have solved the problem by putting $\partial u/\partial x = p$ and following the same method thereafter. We would then have arrived at the general solution

$$u(x, y) = h(y) + \int g_1(x),$$

where h, g_1 are arbitrary. It follows from this that we can express the general solution most simply as

$$u(x, y) = g(x) + h(y).$$

Since h is a primitive of f and g is a primitive of g_1, both h and g must be differentiable, but are otherwise arbitrary.

Solution to Exercise 2

Putting $\partial u/\partial x = p$, the equation becomes

$$\boxed{\frac{\partial p}{\partial x} = 0},$$

with solution

$$p(x, y) = \boxed{f(y)},$$

for arbitrary f. Since the equation

$$\frac{\partial u}{\partial x} = f(y)$$

has solution

$$u = xf(y) + g(y),$$

we have for the general solution of the original equation

$$u(x, y) = \boxed{g(y) + xf(y)},$$

where f and g are arbitrary functions of one variable.

Solution to Exercise 3

Putting $\partial u/\partial y = p$, the equation becomes

$$\frac{\partial p}{\partial y} - \frac{1}{y}p = 0.$$

An integrating factor is

$$\exp \int \left(-\frac{1}{y}\right) dy = \exp(-\ln y)$$

$$= \frac{1}{y}$$

(compare with Exercise 5 of subsection 0.2.2), and multiplying through by this factor brings the equation to the form

$$\frac{\partial}{\partial y}\left(\frac{p(x, y)}{y}\right) = 0,$$

which has solution

$$p(x, y) = \boxed{yf(x)},$$

where f is arbitrary. Since $p = \partial u/\partial y$, an integration with respect to y gives

$$u(x, y) = \boxed{\tfrac{1}{2}y^2 f(x) + g(x)}.$$

The factor $\frac{1}{2}$ may without loss of generality be absorbed into the arbitrary function f.

Solution to Exercise 4

In terms of p, where $p = \partial u/\partial y$, the equation is

$$x\frac{\partial p}{\partial x} = 2p,$$

or in normal form,

$$\frac{\partial p}{\partial x} - \frac{2}{x}p = 0.$$

An integrating factor is

$$\exp \int \left(-\frac{2}{x}\right) dx = \frac{1}{x^2},$$

bringing the equation to the form

$$\frac{\partial}{\partial x}\left(\frac{p(x, y)}{x^2}\right) = 0.$$

Integrating with respect to x and solving for p gives

$$p(x, y) = x^2 f(y)$$

with f arbitrary. Since $p = \partial u/\partial y$ we can now integrate with respect to y to obtain

$$u(x, y) = \boxed{x^2 \int f(y) + g(x)}.$$

0S.3.3 Transforming Second-Order Derivatives

Solution to Exercise 1

From Exercise 6 of subsection 0.2.6,

$$\frac{\partial}{\partial x} = \boxed{\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}},$$

$$\frac{\partial}{\partial t} = \boxed{c\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)}.$$

Since $\partial^2/\partial x^2 = (\partial/\partial x)^2$, $\partial^2/\partial x \partial t = (\partial/\partial x)(\partial/\partial t)$ etc., we then have

$$\frac{\partial^2}{\partial x^2} = \boxed{\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}},$$

$$\frac{\partial^2}{\partial x \partial t} = \boxed{c\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2}\right)},$$

$$\frac{\partial^2}{\partial t^2} = \boxed{c^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)}.$$

Solution to Exercise 2

We see from Exercise 1 that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right).$$

Subtracting the second equation from c^2 times the first, we have

$$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 4c^2\frac{\partial^2 u}{\partial \xi \partial \eta},$$

so that

$$\boxed{\frac{\partial^2 u}{\partial \xi \partial \eta}} = 0.$$

By reference to the solution of Exercise 1 in Subsection 0.3.2 we see that the general solution of this equation is

$$u(\xi, \eta) = \boxed{f(\xi) + g(\eta)},$$

for arbitrary differentiable functions f and g, so that the solution of the wave equation is

$$u(x, t) = \boxed{f(x + ct) + g(x - ct)}.$$

Solution to Exercise 3

Using the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}.$$

Since the given expressions for ξ, η mean that

$$\frac{\partial \xi}{\partial x} = \alpha, \frac{\partial \xi}{\partial t} = \beta, \frac{\partial \eta}{\partial x} = \gamma, \frac{\partial \eta}{\partial t} = \delta,$$

we have

$$\frac{\partial}{\partial x} = \alpha\frac{\partial}{\partial \xi} + \gamma\frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = \beta\frac{\partial}{\partial \xi} + \delta\frac{\partial}{\partial \eta}.$$

The second-order derivatives are, therefore,

$$\frac{\partial^2}{\partial x^2} = \alpha^2\frac{\partial^2}{\partial \xi^2} + 2\alpha\gamma\frac{\partial^2}{\partial \xi \partial \eta} + \gamma^2\frac{\partial^2}{\partial \eta^2},$$

$$\frac{\partial^2}{\partial x \partial t} = \alpha\beta\frac{\partial^2}{\partial \xi^2} + (\alpha\delta + \beta\gamma)\frac{\partial^2}{\partial \xi \partial \eta} + \gamma\delta\frac{\partial^2}{\partial \eta^2},$$

$$\frac{\partial^2}{\partial t^2} = \beta^2\frac{\partial^2}{\partial \xi^2} + 2\beta\delta\frac{\partial^2}{\partial \xi \partial \eta} + \delta^2\frac{\partial^2}{\partial \eta^2},$$

so that

$$L = \boxed{\begin{gathered}(A\beta^2 + B\alpha\beta + C\alpha^2)\frac{\partial^2}{\partial \xi^2} + (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma)\frac{\partial^2}{\partial \xi \partial \eta} \\ +(A\delta^2 + B\gamma\delta + C\gamma^2)\frac{\partial^2}{\partial \eta^2}\end{gathered}}$$

Solution to Exercise 4

By either of the ways suggested we get

$$\frac{\partial}{\partial r} = \boxed{\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}},$$

$$\frac{\partial}{\partial \theta} = \boxed{-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}}.$$

Solution to Exercise 5

We have

$$\begin{bmatrix}\dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y}\end{bmatrix}.$$

The matrix of coefficients can be inverted in the usual way:

$$\begin{bmatrix}\cos\theta & \sin\theta & 1 & 0 \\ -r\sin\theta & r\cos\theta & 0 & 1\end{bmatrix}$$

Stage 1: Multiply first row by $\sec\theta$ and then add $r\sin\theta$ times the first row to the second row. This gives

$$\begin{bmatrix} 1 & \tan\theta & \sec\theta & 0 \\ 0 & r\sec\theta & r\tan\theta & 1 \end{bmatrix}.$$

Stage 2: Multiply second row by $(1/r)\cos\theta$ and then subtract $\tan\theta$ times the second row from the first row. This gives

$$\begin{bmatrix} 1 & 0 & \cos\theta & -\dfrac{1}{r}\sin\theta \\ 0 & 1 & \sin\theta & \dfrac{1}{r}\cos\theta \end{bmatrix}.$$

Reading off the inverse of the matrix we started with,

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\dfrac{1}{r}\sin\theta \\ \sin\theta & \dfrac{1}{r}\cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \end{bmatrix},$$

or in other words,

$$\frac{\partial}{\partial x} = \boxed{\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}},$$

$$\frac{\partial}{\partial y} = \boxed{\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}}.$$

Solution to Exercise 6

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right) \\
&\quad + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} \\
&\quad - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}.
\end{aligned}$$

Therefore

$$\frac{\partial^2}{\partial x^2} = \boxed{\cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}}.$$

Similarly,

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \left(\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right) \\
&= \sin^2\theta\frac{\partial^2}{\partial r^2} + \sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\right) \\
&\quad + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\left(\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)
\end{aligned}$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}$$

$$+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Therefore,

$$\frac{\partial^2}{\partial y^2} = \boxed{\sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}}.$$

It follows that

$$\nabla^2 = \boxed{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}.$$

Solution to Exercise 7

(i) Since

$$\frac{\partial\xi}{\partial x} = 2x, \frac{\partial\eta}{\partial x} = 0, \frac{\partial\xi}{\partial t} = 2t, \frac{\partial\eta}{\partial t} = 1,$$

we have by the chain rule,

$$\frac{\partial}{\partial x} = \boxed{2x\frac{\partial}{\partial\xi}},$$

$$\frac{\partial}{\partial t} = \boxed{2t\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}}.$$

(ii) From the first answer in part (i), we have

$$\frac{\partial}{\partial\eta} = \boxed{\frac{1}{2x}\frac{\partial}{\partial x}}.$$

Substituting this into the second answer in part (i) gives

$$\frac{\partial}{\partial\eta} = \boxed{\frac{\partial}{\partial t} - \frac{t}{x}\frac{\partial}{\partial x}}.$$

(iii)

$$\frac{\partial^2}{\partial\eta^2} = \left(\frac{\partial}{\partial t} - \frac{t}{x}\frac{\partial}{dx}\right)\left(\frac{\partial}{\partial t} - \frac{t}{x}\frac{\partial}{\partial x}\right)$$

$$= \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\left(\frac{t}{x}\frac{\partial}{\partial x}\right) - \frac{t}{x}\frac{\partial^2}{\partial x\partial t} + \frac{t}{x}\frac{\partial}{\partial x}\left(\frac{t}{x}\frac{\partial}{\partial x}\right)$$

$$= \frac{\partial^2}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x} - \frac{t}{x}\frac{\partial^2}{\partial x\partial t} - \frac{t}{x}\frac{\partial^2}{\partial x\partial t} - \frac{t^2}{x^3}\frac{\partial}{\partial x} + \frac{t^2}{x^2}\frac{\partial^2}{\partial x^2},$$

so that we have

$$\frac{\partial^2}{\partial\eta^2} = \frac{1}{x^2}L - \boxed{\frac{1}{x}\left(1 + \frac{t^2}{x^2}\right)\frac{\partial}{\partial x}}.$$

(iv) The last equation in part (iii) can be written as

$$L = x^2\frac{\partial^2}{\partial\eta^2} + x\left(1 + \frac{t^2}{x^2}\right)\frac{\partial}{\partial x},$$

and using the first answer in part (i) gives

$$L = \boxed{x^2\frac{\partial^2}{\partial\eta^2} + 2(x^2 + t^2)\frac{\partial}{\partial\xi}}.$$

Solution to Exercise 8

Using the solution to Exercise 7(i), we see that the right-hand side of the given equation can be expressed as

$$\frac{x^2}{t}\left(2t\frac{\partial u}{\partial\xi} + \frac{\partial u}{\partial\eta}\right) + \frac{t^2}{x}\left(2x\frac{\partial u}{\partial\xi}\right) = 2(x^2 + t^2)\frac{\partial u}{\partial\xi} + \frac{x^2}{t}\frac{\partial u}{\partial\eta},$$

so that upon using the expression for L derived in Exercise 7(iv), dividing through by x^2 and replacing t by η, the transformed equation is seen to be

$$\boxed{\frac{\partial^2 u}{\partial\eta^2} = \frac{1}{\eta}\frac{\partial u}{\partial\eta}}.$$

By reference to Exercise 3 of subsection 0.3.2 we see that the general solution of the transformed equation is

$$u(\xi, \eta) = \boxed{f(\xi) + \eta^2 g(\xi)},$$

where f and g are arbitrary functions of one variable. Thus the general solution of the original equation is

$$u(x, t) = \boxed{f(x^2 + t^2) + t^2 g(x^2 + t^2)}.$$

Solution to Exercise 9

In Exercise 7 of subsection 0.1.1 we saw that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2},$$

and using this result in conjunction with the composite function rule we get

$$\frac{d}{dx}(\tfrac{1}{2}\arctan\tfrac{1}{2}x) = \frac{1}{4 + x^2},$$

so that we have

$$\frac{\partial\xi}{\partial x} = \frac{1}{1 + x^2}, \quad \frac{\partial\xi}{\partial t} = -1,$$

$$\frac{\partial\eta}{\partial x} = \frac{1}{4 + x^2}, \quad \frac{\partial\eta}{\partial t} = -1.$$

The chain rule then gives

$$\frac{\partial}{\partial x} = \frac{1}{1 + x^2}\frac{\partial}{\partial\xi} + \frac{1}{4 + x^2}\frac{\partial}{\partial\eta},$$

$$\frac{\partial}{\partial t} = -\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right).$$

Solving these equations for $\partial/\partial\xi$ and $\partial/\partial\eta$ we find

$$\frac{\partial}{\partial\xi} = \tfrac{1}{3}(1 + x^2)(4 + x^2)\left(\frac{\partial}{\partial x} + \frac{1}{4 + x^2}\frac{\partial}{\partial t}\right),$$

$$\frac{\partial}{\partial\eta} = -\tfrac{1}{3}(1 + x^2)(4 + x^2)\left(\frac{\partial}{\partial x} + \frac{1}{1 + x^2}\frac{\partial}{\partial t}\right).$$

It follows that

$$\frac{\partial^2}{\partial\xi\partial\eta} = -\tfrac{1}{9}(1 + x^2)(4 + x^2)\left(\frac{\partial}{\partial x} + \frac{1}{4 + x^2}\frac{\partial}{\partial t}\right)$$

$$\times\left\{(1 + x^2)(4 + x^2)\left(\frac{\partial}{\partial x} + \frac{1}{1 + x^2}\frac{\partial}{\partial t}\right)\right\}$$

$$= -\tfrac{1}{9}(1 + x^2)(4 + x^2)\left\{\frac{\partial^2}{\partial t^2} + (5 + 2x^2)\frac{\partial^2}{\partial x\partial t} + (1 + x^2)(4 + x^2)\frac{\partial^2}{\partial x^2}\right.$$

$$\left. + 2x(5 + 2x^2)\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial t}\right\}$$

$$= -\tfrac{1}{9}(1 + x^2)(4 + x^2)\left\{L + 2x\left[(5 + 2x^2)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\right\}.$$

Using the expressions above for $\partial/\partial x$, $\partial/\partial t$ in terms of $\partial/\partial\xi$, $\partial/\partial\eta$,

$$(5 + 2x^2)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = \frac{4 + x^2}{1 + x^2}\frac{\partial}{\partial\xi} + \frac{1 + x^2}{4 + x^2}\frac{\partial}{\partial\eta},$$

so that the expression for L is

$$L = \boxed{\frac{-9}{(1 + x^2)(4 + x^2)}\left[\frac{\partial^2}{\partial\xi\partial\eta} + \frac{2x}{9}(4 + x^2)^2\frac{\partial}{\partial\xi} + \frac{2x}{9}(1 + x^2)^2\frac{\partial}{\partial\eta}\right]},$$

which is equivalent to the formula for $L[u]$ given at the top of page 46 in *W*.

0S.4 VECTOR ALGEBRA AND LINE INTEGRALS

0S.4.1 Vector Algebra

Solution to Exercise 1

(i) If we complete the parallelogram which has (arrows belonging to) $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides, then the diagonal from the point O where the sides $\mathbf{a}$ and $\mathbf{b}$ meet is $\mathbf{a} + \mathbf{b}$. Since the diagonals of a parallelogram bisect each other, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ will be the position vector of a point halfway between the two vertices of the parallelogram with position vectors $\mathbf{a}$ and $\mathbf{b}$ (taking O as the origin). The answer is therefore

$$\boxed{\tfrac{1}{2}(\mathbf{a} + \mathbf{b})}\,.$$

(ii) The answer here depends on how the arrows are arranged. If $\mathbf{b}$ points from the end of the arrow $\mathbf{a}$ (or vice versa) then the third side of the triangle is

$$\boxed{\mathbf{a} + \mathbf{b}}\,,$$

or $-\mathbf{a} - \mathbf{b}$, depending upon how the vector is directed along the side. If $\mathbf{a}$ and $\mathbf{b}$ have corresponding ends coincident then the third side of the triangle will be $\mathbf{a} - \mathbf{b}$ or $\mathbf{b} - \mathbf{a}$, depending on what direction is associated with it.

(iii) Since by the definition of the scalar product,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, the required cosine is equal to

$$\boxed{\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}}\,.$$

(iv) The projection of $\mathbf{a}$ along the direction of $\mathbf{b}$ is

$$|\mathbf{a}| \cos\theta,$$

which, we see from part (iii), is equal to

$$\boxed{\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}}\,.$$

(v) Since $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ the required unit vector is

$$\boxed{\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}}\,.$$

(vi) The parallelogram's area is $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$, but we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ by definition. The area is therefore

$$\boxed{|\mathbf{a} \times \mathbf{b}|}\,.$$

(vii) $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if $\theta = \pi/2$, or $\cos\theta = 0$, or

$$\boxed{\mathbf{a} \cdot \mathbf{b} = 0}\,.$$

(viii) $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\theta = 0$, or $\sin\theta = 0$, or

$$\boxed{\mathbf{a} \times \mathbf{b} = \mathbf{0}}\,.$$

Solution to Exercise 2

(i) If $\mathbf{a} = \mathbf{i} + \mathbf{j}$, then $|\mathbf{a}| = \sqrt{1^2 + 1^2}$, or,

$$|\mathbf{a}| = \boxed{\sqrt{2}}.$$

(ii)

$$\mathbf{a} - 2\mathbf{b} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

so that

$$|\mathbf{a} - 2\mathbf{b}| = \sqrt{3^2 + 3^2 + 2^2},$$

or

$$|\mathbf{a} - 2\mathbf{b}| = \boxed{\sqrt{22}}.$$

(iii)

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2 - 1,$$

so that

$$\mathbf{a} \cdot \mathbf{b} = \boxed{1}.$$

(iv)

$$\mathbf{a} \times \mathbf{b} = (\mathbf{i} + \mathbf{j}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k}).$$

Using the formulae for vector products involving the unit vectors, **i**, **j** and **k** (see page 8 of *SS* if necessary) we see that this is equal to

$$-\mathbf{i} \times \mathbf{j} - \mathbf{i} \times \mathbf{k} + \mathbf{j} \times 2\mathbf{i} - \mathbf{j} \times \mathbf{k} = -\mathbf{k} + \mathbf{j} - 2\mathbf{k} - \mathbf{i},$$

so that

$$\mathbf{a} \times \mathbf{b} = \boxed{-\mathbf{i} + \mathbf{j} - 3\mathbf{k}}.$$

(v)

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = (\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}).$$

Proceeding as in part (iv),

$$\begin{aligned}
(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})
&= 2\mathbf{i} \times 3\mathbf{j} + 2\mathbf{i} \times \mathbf{k} - \mathbf{j} \times (-\mathbf{i}) - \mathbf{j} \times \mathbf{k} - \mathbf{k} \times (-\mathbf{i}) - \mathbf{k} \times 3\mathbf{j} \\
&= 6\mathbf{k} - 2\mathbf{j} - \mathbf{k} - \mathbf{i} + \mathbf{j} + 3\mathbf{i} \\
&= 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}.
\end{aligned}$$

Then

$$(\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) = 2 - 1,$$

so that

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \boxed{1}.$$

Solution to Exercise 3

(i) The velocity $\mathbf{v}$ of the particle is equal to $d\mathbf{r}/dt$. This is

$$\boxed{-6 \sin 3t\mathbf{i} + 6 \cos 3t\mathbf{j} + 2\mathbf{k}}.$$

(ii) The speed is the magnitude of the velocity. Since

$$|\mathbf{v}| = \sqrt{36 \sin^2 3t + 36 \cos^2 3t + 4},$$

the speed is

$$\boxed{\sqrt{40}}.$$

(iii) Since the particle is travelling in the direction of its velocity vector at any instant, the unit vector required is $\mathbf{v}/|\mathbf{v}|$. From parts (i) and (ii) this is

$$\boxed{\frac{1}{\sqrt{40}}(-6\sin 3t\mathbf{i} + 6\cos 3t\mathbf{j} + 2\mathbf{k})}.$$

0S.4.2 Line Integrals

Solution to Exercise 1

Since

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} &= (x\mathbf{i} + x^2y\mathbf{j})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy)\\ &= x\,dx + x^2y\,dy,\end{aligned}$$

and since $dy = 0$ on the line segment from $(-1, 0)$ to $(1, 0)$,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-1}^{1} x\,dx\\ &= [\tfrac{1}{2}x^2]_{-1}^{1},\end{aligned}$$

so that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \boxed{0}.$$

Solution to Exercise 2

Clearly, from the answer to Exercise 1, the integral is equal to the integral along the line segment from $(1, 0)$ to $(1, 1)$. Since $x = 1$ and $dx = 0$ here,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{0}^{1} y\,dy\\ &= [\tfrac{1}{2}y^2]_0^1,\end{aligned}$$

so that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \boxed{\tfrac{1}{2}}.$$

Solution to Exercise 3

Here we have

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2$$

along C, so that

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C x\,dx + \int_C x^2y\,dy\\ &= \int_0^1 t^2\cdot 2t\,dt + \int_0^1 t^4\cdot t^3\cdot 3t^2\,dt\\ &= \int_0^1 (2t^3 + 3t^9)\,dt\\ &= [\tfrac{1}{2}t^4 + \tfrac{3}{10}t^{10}]_0^1,\end{aligned}$$

so that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \boxed{\tfrac{4}{5}}.$$

Solution to Exercise 4

The semicircle is given in Cartesian coordinates by

$$x^2 + y^2 = 1, y \geqslant 0.$$

We see that $x^2 + y^2$ will be equal to unity if we put

$$x = \cos t, \ y = \sin t,$$

and that t will vary from 0 to π as the point (x, y) moves around the semicircle from $(1, 0)$ to $(-1, 0)$. Thus our parametric representation of the semicircle is

$$x = \boxed{\cos t},$$

$$y = \boxed{\sin t},$$

$$t \in \boxed{[0, \pi]}.$$

Solution to Exercise 5

Here we have

$$\mathbf{F} \cdot d\mathbf{r} = (\mathbf{i} + y\mathbf{j}) \cdot (\mathbf{i}\, dx + \mathbf{j}\, dy)$$
$$= dx + y\, dy,$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C dx + \int_C y\, dy.$$

On the semicircle,

$$\frac{dx}{dt} = -\sin t, \ \frac{dy}{dt} = \cos t,$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^\pi \sin t\, dt + \int_0^\pi \sin t \cos t\, dt$$
$$= [\cos t - \tfrac{1}{4}\cos 2t]_0^\pi,$$

or

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{-2}.$$

Solution to Exercise 6

We have

$$\mathbf{v} \times d\mathbf{r} = (xy\, dy - y^2\, dx)\mathbf{k},$$

so that using the parametrization of Exercise 4 for the semicircle,

$$\int_C \mathbf{v} \times d\mathbf{r} = \left\{\int_C xy\, dy - \int_C y^2\, dx\right\}\mathbf{k}$$
$$= \left\{\int_0^\pi \cos t \sin t \cos t\, dt - \int_0^\pi \sin^2 t(-\sin t)\, dt\right\}\mathbf{k}$$
$$= \left\{\int_0^\pi \sin t\, dt\right\}\mathbf{k}$$
$$= [-\cos t]_0^\pi\mathbf{k},$$

so that

$$\int_C \mathbf{v} \times d\mathbf{r} = \boxed{2\mathbf{k}}.$$

Since

$$\int_C \mathbf{v} \times d\mathbf{r} = \mathbf{k}\int_C \mathbf{v} \cdot \mathbf{n}\, dr,$$

it follows that

$$\int_C \mathbf{v} \cdot \mathbf{n}\, dr = \boxed{2}.$$

0S.5 DOUBLE AND VOLUME INTEGRALS

0S.5.1 Double Integrals

Solution to Exercise 1

The region of integration is enclosed between the lines $x = 0$, $x = 1$, $y = 0$ and $y = 2x + 1$. A diagram of this region can be found on page 17 of *Unit MST 282 6*, or on page 11 of *Unit 3*.

Solution to Exercise 2

$$\int_0^{2x+1} xy\,dy = x\int_0^{2x+1} y\,dy$$
$$= x[\tfrac{1}{2}y^2]_0^{2x+1},$$

that is,

$$\int_0^{2x+1} xy\,dy = \boxed{\tfrac{1}{2}x(2x+1)^2}.$$

Solution to Exercise 3

$$\int_0^1 \tfrac{1}{2}x(2x+1)^2\,dx = \tfrac{1}{2}\int_0^1 (4x^3 + 4x^2 + x)\,dx$$
$$= \tfrac{1}{2}[x^4 + \tfrac{4}{3}x^3 + \tfrac{1}{2}x^2]_0^1$$
$$= \tfrac{1}{2}[1 + \tfrac{4}{3} + \tfrac{1}{2}],$$

so that

$$\int_0^1\left[\int_0^{2x+1} xy\,dy\right]dx = \boxed{\tfrac{17}{12}}.$$

Solution to Exercise 4

The "inner" integration is performed with respect to y. Since the ellipse can be written as

$$y = \pm b\sqrt{\left(1 - \frac{x^2}{a^2}\right)},$$

the region enclosed by the ellipse can be described by

$$-b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \leqslant y \leqslant b\sqrt{\left(1 - \frac{x^2}{a^2}\right)},$$
$$-a \leqslant x \leqslant a.$$

The limits (end-points) of the two integrations are then the upper and lower bounds in these two inequalities. Therefore:

$$\text{Limits of } y\text{-integration are}\begin{cases}\boxed{b\sqrt{(1 - x^2/a^2)}} \\ \boxed{-b\sqrt{(1 - x^2/a^2)}}\end{cases};$$

$$\text{Limits of } x\text{-integration are}\begin{cases}\boxed{a} \\ \boxed{-a}\end{cases};$$

Resulting double integral is

$$\boxed{\int_{-a}^{a}\left[\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dy\right]dx}\,.$$

Solution to Exercise 5

$$\begin{aligned}\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}&\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dy\\ &=\left[\left(1-\frac{x^2}{a^2}\right)y-\frac{y^3}{3b^2}\right]_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}\\ &=\left[1-\frac{x^2}{a^2}-\tfrac{1}{3}\left(1-\frac{x^2}{a^2}\right)\right]2b\sqrt{\left(1-\frac{x^2}{a^2}\right)},\end{aligned}$$

so that

Inner integral is $\boxed{\tfrac{4}{3}b\left(1-\dfrac{x^2}{a^2}\right)^{\frac{3}{2}}}$.

We are left with the integral

$$\int_{-a}^{a}\tfrac{4}{3}b\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx$$

to evaluate. The substitution to make here is $x = a\sin v$. (You may like to refer to section III.3 of *TI* to see why we choose this particular substitution. We can see that our integral is of the form given at the beginning of that section, with $f(x) = 0$, $h(x) = \frac{4}{3}b(1 - x^2/a^2)$, $ax^2 + bx + c$ replaced by $1 - x^2/a^2$, $g(x) = 1$, $k(x) = 0$. Subsection III.3.2(iii) gives the substitution we are using.) The range $-a \leqslant x \leqslant a$ is equivalent to $-\pi/2 \leqslant v \leqslant \pi/2$, and $dx/dv = a\cos v$. Thus the integral becomes

$$\tfrac{4}{3}ba\int_{-\pi/2}^{\pi/2}\cos^4 v\,dv = \tfrac{1}{2}\pi ab,$$

using the information given in the question. Thus:

Final result is $\boxed{\tfrac{1}{2}\pi ab}$.

Solution to Exercise 6

Putting $x^2 + y^2 = r^2$ and replacing $dx\,dy$ by $r\,dr\,d\theta$,

$$\iint_D\left(1-\frac{x^2+y^2}{a^2}\right)dx\,dy = \iint_D\left(1-\frac{r^2}{a^2}\right)r\,dr\,d\theta.$$

D is composed of points (r, θ), where

$$0 \leqslant r \leqslant a,$$

$$-\pi \leqslant \theta \leqslant \pi.$$

Therefore, the double integral in polar coordinates is

$$\boxed{\int_{0}^{a}\left[\int_{-\pi}^{\pi}\left(1-\frac{r^2}{a^2}\right)r\,d\theta\right]dr}\,.$$

The inner integral is equal to

$$\left(1-\frac{r^2}{a^2}\right)r\int_{-\pi}^{\pi}d\theta = 2\pi r\left(1-\frac{r^2}{a^2}\right).$$

The outer integral is

$$2\pi\int_0^a \left(r - \frac{r^3}{a^2}\right) dr = 2\pi\left[\tfrac{1}{2}r^2 - \frac{r^4}{4a^2}\right]_0^a$$
$$= 2\pi[\tfrac{1}{2}a^2 - \tfrac{1}{4}a^2],$$

so that the value of the integral is

$$\boxed{\tfrac{1}{2}\pi a^2}.$$

This agrees with the result of Exercise 5 when $b = a$.

0S.5.2 Volume Integrals

Solution to Exercise 1

The interior of the cube consists of points (x, y, z) for which

$$0 \leqslant x \leqslant 1,$$
$$0 \leqslant y \leqslant 1,$$
$$0 \leqslant z \leqslant 1.$$

Thus the triple integral is

$$\boxed{\int_0^1 \left\{\int_0^1 \left[\int_0^1 (x + y + z)\,dx\right] dy\right\} dz}.$$

Since

$$\int_0^1 (x + y + z)\,dx = [\tfrac{1}{2}x^2 + (y + z)x]_0^1,$$

the inner integral is

$$\boxed{\tfrac{1}{2} + y + z}.$$

Since

$$\int_0^1 (\tfrac{1}{2} + y + z)\,dy = [(\tfrac{1}{2} + z)y + \tfrac{1}{2}y^2]_0^1,$$

the next integral is

$$\boxed{1 + z}.$$

Since

$$\int_0^1 (1 + z)\,dz = [z + \tfrac{1}{2}z^2]_0^1,$$

the outer integral is

$$\boxed{\tfrac{3}{2}}.$$

Solution to Exercise 2

The interior of the prism consists of points (x, y, z) for which

$$0 \leqslant x \leqslant 1$$
$$0 \leqslant y \leqslant 1 - z$$
$$0 \leqslant z \leqslant 1,$$

so that the triple integral is

$$\int_{\boxed{0}}^{\boxed{1}} \left\{ \int_{\boxed{0}}^{\boxed{1 - z}} \left[\int_{\boxed{0}}^{\boxed{1}} 3x^2yz\,dx \right] dy \right\} dz.$$

Solution to Exercise 3

$$\int_0^1 3x^2yz\,dx = [x^3yz]_0^1,$$

so that the inner integral is

$$\boxed{yz}\,.$$

$$\int_0^{1-z} yz\,dy = [\tfrac{1}{2}y^2z]_0^{1-z},$$

so that the next integral is

$$\boxed{\tfrac{1}{2}z(1 - z)^2}\,.$$

$$\tfrac{1}{2}\int_0^1 z(1 - z)^2\,dz = \tfrac{1}{2}\int_0^1 (z - 2z^2 + z^3)dz$$
$$= \tfrac{1}{2}[\tfrac{1}{2}z^2 - \tfrac{2}{3}z^3 + \tfrac{1}{4}z^4]_0^1\,,$$

so that the outer integral is

$$\boxed{\tfrac{1}{24}}\,.$$

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